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SIT–NLS integrable equations associated with Hermitian symmetric spaces

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Abstract

We study multi-component SIT–NLS systems associated with Hermitian symmetric spaces. We introduce generalized polarizations and express them in terms of the variable of Hermitian symmetric spaces. The equations of motion are expressed using electromagnetic fields and generalized polarizations. The Bäcklund transformation of the SIT–NLS system is introduced, which relates two solutions of SIT–NLS systems.

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1. Introduction

The phenomenon of self-induced transparency (SIT) is due to the fact that optical solitons occur as a result of coherent interaction of the optical pulse with the resonant atoms like erbium. On the other hand, the nonlinear Schrödinger (NLS) equation describes the lossless propagation of optical solitons along a fibre when the dispersion gets balanced by the nonlinearity of the medium. The possibility of coexistence of SIT–NLS solitons was first reported in [1], while its detailed physical mechanism was analysed by Nakazawa *et al* [2]. They note that SIT offers the possibility of pulse shaping and standardization that is different from the NLS soliton formation. Since then the SIT–NLS system has been thoroughly analysed, including the integrability properties like Painlevé analysis [3], Lax pair, soliton solutions [4], and their inhomogeneous generalization, etc [5]. In these studies, the SIT–NLS soliton system describes the propagation of optical solitons in an optical fibre doped with two-level resonant atoms. These kinds of fibres can be used, for example, for the all-optical communication system by manipulating the effect of SIT.

The extension of the SIT–NLS system to the case of multi-level resonant atoms is interesting. It can be used to describe important phenomena related with the SIT effect of multi-level resonant atoms like soliton cloning [6]. In fact, [7] describes a three-level

SIT–NLS system, used to study soliton clone generation. The equation used for this system was

$$
\begin{aligned}
\frac{\partial A_{12}}{\partial x} &= -\frac{\mathrm{i}}{2}\beta_{22}\frac{\partial^2 A_{12}}{\partial \tau^2} + \mathrm{i}\gamma_{12}[|A_{12}|^2 + 2|A_{13}|^2]A_{12} + \frac{\mathrm{i}\omega_{12}n_a}{2\epsilon_0 c}c_2^* c_1 \mu_{12}, \\
\frac{\partial A_{13}}{\partial x} &= -\frac{\mathrm{i}}{2}\beta_{23}\frac{\partial^2 A_{13}}{\partial \tau^2} + \mathrm{i}\gamma_{13}[|A_{13}|^2 + 2|A_{12}|^2]A_{13} + \frac{\mathrm{i}\omega_{13}n_a}{2\epsilon_0 c}c_3^* c_1 \mu_{12}, \\
\frac{\partial c_1}{\partial t} &= \frac{\mathrm{i}}{\hbar}[c_2\mu_{12}A_{12}(t) + c_3\mu_{13}A_{13}(t)], \\
\frac{\partial c_2}{\partial t} &= \frac{\mathrm{i}}{\hbar}[c_1\mu_{12}A_{12}^*(t)], \qquad \frac{\partial c_3}{\partial t} = \frac{\mathrm{i}}{\hbar}[c_1\mu_{13}A_{13}^*(t)],
\end{aligned}
\tag{1}
$$

where A_{ij} are the optical pulses tunes in a frequency ω_{ij}, while c_i denote the probability amplitude of the atomic level i, and $\tau = (t - x/v_g)/T_0$ with T_0 as the pulse width and v_g as the average group velocity. For other notation, see [7]. Equation (1) describes the coherent interaction of optical solitons with three-level atoms, where c_1 is the probability amplitude of the ground state, while c_2, c_3 are those of the excited states. This equation becomes integrable for special values of the parameters μ_i, β_i, γ_i and for special forms of triple interaction terms like $|A_i|^2 A_j$.

The integrability property of a nonlinear equation is important in studying the physical properties of the system. They admit the use of the inverse scattering method, permitting analytical solutions including multi-soliton solutions. Considering the applicability of the SIT–NLS solitons, it is desired to have integrable SIT–NLS equations of multi-level resonant atoms. One possible scheme in this direction is to use multi-level generalizations based on Hermitian symmetric spaces (HSS). It is well known that the Hermitian symmetric spaces admit multi-level generalizations of various nonlinear equations [8] including the NLS equation [9, 10], the derivative NLS equation [11], the KdV and mKdV equations [12]. Even though not explicit, the Hermitian symmetric space also appeared in describing multi-level SIT systems [13].

In this paper, we construct multi-level SIT–NLS systems based on the Hermitian symmetric spaces. In this formalism, the group elements g of the HSS are related to the probability amplitudes of atomic levels, while components of $g^{-1}\partial g$ correspond to the optical pulses A_{ij}. In the case of the two-level SIT–NLS system, the probability amplitudes c_i are rewritten in the form of $D = |c_2|^2 - |c_1|^2$ and $P = c_1 c_2^*$ by using the population inversion variable D and the polarization P. This makes it possible to understand intuitively the way in which the phase rotation of the dipole changes. In the case of the multi-level SIT–NLS system, a similar formulation is possible using generalized variables for polarizations and population inversions, which are named *generalized polarizations*. They are related with the group elements of the HSS by $g^{-1}\bar{T}g$. These variables are related to the density matrix; see more details and their physical meanings in [14] for the SIT case.

The Lax pair of multi-level SIT–NLS systems are constructed in terms of the group elements g, and the equation of motion for g is obtained by requiring the compatibility of the Lax pairs. We then discuss the Bäcklund transformation (BT) in two forms [15, 16]. The BT is important as it gives multi-soliton solutions, avoiding the mathematical technicalities of the inverse scattering method. It is also related with the problem of integrable boundary conditions [17, 18].

Section 2 introduces Hermitian symmetric spaces. Some specific features of the HSS are explained, which are essential for the construction of the multi-level SIT–NLS system. Explicit examples of the HSS are given in section 3. Their equations of motion are expressed in terms of the optical pulses $g^{-1}\partial g$ and generalized polarizations $g^{-1}\bar{T}g$. BTs in two forms, type-I

BT and type-II BT, are explained in section 4. The relation between the BT and equations of motion is also explained in section 4. These BTs will be used in a separate paper [19] in constructing explicit one- and two-soliton solutions of SIT–NLS systems associated with the HSSs. The appendix proves equation (31) in section 4.

2. The SIT–NLS system in Hermitian symmetric spaces

2.1. The Hermitian symmetric space

The complex structure of the Hermitian symmetric space G/K [20] is essential in constructing the multi-component SIT–NLS system. Let **g** and **k** be the associated Lie algebras for G/K whose orthogonal decomposition, $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$, satisfies the commutation relations,

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}, \qquad [\mathbf{k}, \mathbf{m}] \subset \mathbf{m}, \qquad [\mathbf{m}, \mathbf{m}] \subset \mathbf{k}. \tag{2}$$

The Hermiticity of G/K implies that there exists an element T in the Cartan subalgebra of **k** whose adjoint action defines a complex structure such that $[T, [T, m]] = -m$ for $m \in \mathbf{m}$. The subalgebra **k** is characterized by the property that it commutes with T, i.e. $[T, k] = 0$ for $k \in \mathbf{k}$.

The complete classification of Hermitian symmetric spaces is known [20] in terms of four series and two exceptional cases; $AIII = SU(m+n)/(SU(m) \times SU(n) \times U(1)), CI = Sp(n)/U(n), DIII = SO(2n)/U(n), BDI = SO(m+2)/(SO(m) \times SO(2)), EIII = E_6/(SO(10) \times SO(2)), EVII = E_7/(E_6 \times SO(2))$. In this paper, we treat the following cases of SIT–NLS systems; (1) $AIII$, (2) CI with $n = 2$, i.e., $Sp(2)/U(2)$, (3) $DIII$ with $n = 4$, i.e. $SO(8)/U(4)$.

2.2. The Lax pair

The Lax pair of the SIT–NLS equation is given by

$$\begin{aligned} 0 &= L_z\Psi \equiv [\partial + E + \lambda T]\Psi, \\ 0 &= L_{\bar z}\Psi \equiv \left[\bar\partial + \frac{1}{2}[E, \tilde E] - \partial\tilde E - \lambda E - \lambda^2 T - \frac{1}{\lambda} g^{-1}\bar T g\right]\Psi. \end{aligned} \tag{3}$$

Here $\partial \equiv \partial/\partial\tau, \bar\partial \equiv \partial/\partial x, \bar T = -\alpha T$ with a constant α, $E \equiv g^{-1}\partial g \in \mathbf{m}$ and $\tilde E \equiv [T, E] \in \mathbf{m}$. E denotes the optical pulses which corresponds to A_{ij} in equation (1), while $g^{-1}\bar T g$ describes the generalized polarization. At the present stage, g is the independent variable of the theory.

We now study the compatibility condition of over-determined linear equations; $[L_z, L_{\bar z}] = 0$, which should hold for all values of λ. The compatibility conditions at $O(\lambda^3)$ and $O(\lambda^2)$ are trivial, while at $O(\lambda^1)$ it becomes

$$-\partial E + \tfrac{1}{2}[T, [E, [T, E]]] - [T, [T, \partial E]] = 0. \tag{4}$$

Here, we note that $[T, [T, \partial E]] = -\partial E$ as $\partial E \in \mathbf{m}$ and $[T, [E, [T, E]]] = 0$ as $[T, k] = 0$ for $k \in \mathbf{k}$, which prove equation (4). At $O(\lambda^0)$, the compatibility condition becomes the equation of motion of the SIT–NLS system,

$$\bar\partial E = -\partial^2\tilde E + \tfrac{1}{2}[E, [E, \tilde E]] - [T, g^{-1}\bar T g]. \tag{5}$$

At $O(\lambda^{-1})$, the compatibility condition becomes an identity[1]

$$\partial(g^{-1}\bar T g) + [E, g^{-1}\bar T g] = 0. \tag{6}$$

We note that the property $E \equiv g^{-1}\partial g \in \mathbf{m}$ is essential for the compatibility of the overdetermined Lax pair, leading to the integrable SIT–NLS equation.

[1] $\partial g^{-1} = -g^{-1}\partial g g^{-1}$.

3. Equation of motion

Equation (5) is an equation for the variable $g \in G$, where E is related to g as $E = g^{-1}\partial g$. It can be casted in a more familiar form found in equation (1) by taking g and E as two independent variables. In this formalism, the SIT–NLS equations are constituted by two equations, equation (5) and an *auxiliary* equation

$$\partial g = gE. \tag{7}$$

Then, equation (5) corresponds to the first two equation of (1), while equation (7) corresponds to the last two equation of (1).

For *AIII*, CI and *DIII* series, $(m+n) \times (m+n)$ matrices $E \in \mathbf{m}$ and T can be rewritten in block forms,

$$E = \begin{pmatrix} 0 & E_m \\ -E_m^\dagger & 0 \end{pmatrix}, \qquad T = \frac{\mathrm{i}}{2}\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix} \tag{8}$$

where † denotes a Hermitian conjugate and I_n is an $\times n$ identity matrix. The matrix E_m is a complex $n \times m$ matrix for *AIII*, a complex symmetric $n \times n$ matrix $(m = n)$ for CI, and a complex anti-symmetric $n \times n$ matrix $(m = n)$ for *DIII*.

3.1. Equation of $\frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}$ HSS

This model corresponds to taking $G = SU(m+n)$, $K = SU(m) \times SU(n) \times U(1)$, and E and T matrices are given by $(m+n) \times (m+n)$ matrices in equation (8) with

$$E_m = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,m} \\ & \cdots & & \\ \psi_{n,1} & \psi_{n,2} & \cdots & \psi_{n,m} \end{pmatrix}. \tag{9}$$

In fact, the tracelessness of T requires a term $\mathrm{i}\frac{m-n}{2(m+n)}I$ in equation (8), but this does not introduce any difference in the following. Then equation (5) becomes

$$\bar\partial\psi_{i,j} = -\mathrm{i}\partial^2\psi_{i,j} - 2\mathrm{i}\sum_{l=1,n,k=1,m}\psi^*_{l,k}\psi_{l,j}\psi_{i,k} - \alpha\sum_{l=1,n} g^*_{l,i}g_{l,n+j}, \quad i = 1, n,\ j = 1, m. \tag{10}$$

The auxiliary equation (7) becomes

$$\begin{aligned} \partial g_{i,j} &= -\sum_{l=1,m} g_{i,n+l}\psi^*_{j,l}, \qquad && i = 1, n,\ j = 1, n \\ \partial g_{i,n+j} &= \sum_{l=1,n} g_{i,l}\psi_{l,j}, && i = 1, n,\ j = 1, m. \end{aligned} \tag{11}$$

We note that equations (10), (11) reduce to equation (1) by taking $n = 1, m = 2$ and substituting $A_{12} \to \mathrm{i}\psi_{1,1}$, $A_{13} \to \mathrm{i}\psi_{1,2}$, $c_1 \to g^*_{1,1}$, $c_2 \to g^*_{1,2}$, $c_3 \to g^*_{1,3}$.[2]

The generalized polarizations are given by

$$g^{-1}\bar T g \equiv -\begin{pmatrix} D^{(n)} + \frac{\mathrm{i}}{2}\alpha I_n & P \\ -P^\dagger & D^{(m)} - \frac{\mathrm{i}}{2}\alpha I_m \end{pmatrix}. \tag{12}$$

Here, the matrix elements of $D^{(n)}$, $D^{(m)}$ are given by

$$D^{(n)}_{i,j} = \mathrm{i}\alpha\sum_{l=1,n} g^*_{l,i}g_{l,j} - \mathrm{i}\alpha\delta_{i,j}, \qquad D^{(m)}_{i,j} = \mathrm{i}\alpha\sum_{l=1,n} g^*_{l,n+i}g_{l,n+j}, \tag{13}$$

[2] The ration between triple coupling terms of A_{ij} in equation (1) is 1:2, while this reduction gives a ratio of 1:1. It is known that this 1:1 ratio coupling describes the pulse propagation in a randomly birefringent fibre or in an elliptically birefringent fibre. The different form of triple coupling terms can be obtained by choosing different HSSs, which is the motivation of the present work.

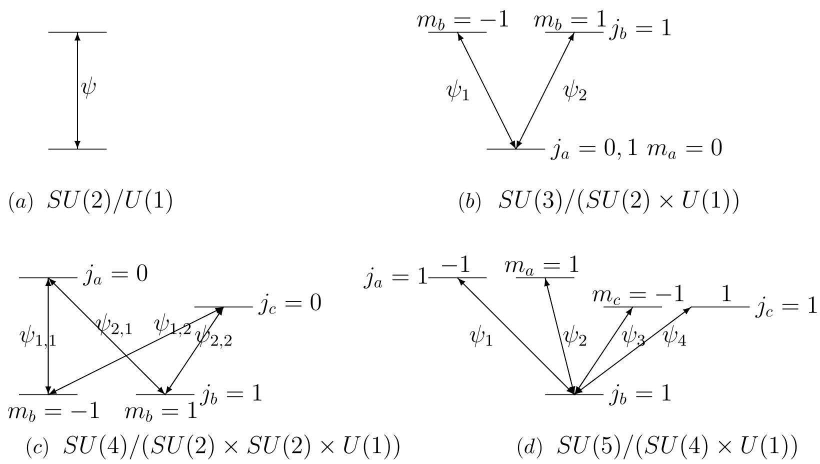

Figure 1. Multi-level resonant systems and their associated Hermitian symmetric spaces.

with the property $D^{(n)\dagger} = -D^{(n)}$ and $D^{(m)\dagger} = -D^{(m)}$. P is an $n \times m$ complex matrix with

$$P_{i,j} = \mathrm{i}\alpha \sum_{l=1,n} g^*_{l,i} g_{l,n+j}. \tag{14}$$

Note that the last term of equation (10) is the generalized polarization, $iP_{i,j}$. Equation (11) can be rewritten in terms of the generalized polarizations. For this, we can use equation (6) to obtain

$$\begin{aligned}
\partial D^{(n)}_{i,j} &= \sum_{l=1,m} (P^*_{j,l}\psi_{i,l} - P_{i,l}\psi^*_{j,l}), && i, j = 1, n,\\
\partial D^{(m)}_{i,j} &= \sum_{l=1,n} (-P^*_{l,i}\psi_{l,j} + P_{l,j}\psi^*_{l,i}), && i, j = 1, m,\\
\partial P_{i,j} &= -\sum_{l=1,m} D^{(m)}_{l,j}\psi_{i,l} + \sum_{l=1,n} D^{(n)}_{i,l}\psi_{l,j}, && i = 1, n,\ j = 1, m.
\end{aligned} \tag{15}$$

Some level structures of resonant atoms in the *AIII* series were known as the W or the M system, described by $SU(5)/(SU(2) \times SU(3) \times U(1))$. There are structures known as Λ or V system, which are described by $SU(3)/(SU(2) \times U(1))$. In figure 1, we show some level structures of resonant atoms and their corresponding HSSs which have been studied in the literature, see [13, 14].

3.2. Equation of $\frac{Sp(2)}{U(2)}$ HSS

This model corresponds to taking $G = Sp(2)$, $K = U(2)$, and E and T matrices are given by 4×4 matrices in equation (8) with $n = m = 2$, and

$$E_m = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{pmatrix}. \tag{16}$$

Then, equation (5) becomes,

$$\begin{aligned}
\bar\partial\psi_1 &= -\mathrm{i}\partial^2\psi_1 - 2\mathrm{i}(|\psi_1|^2 + 2|\psi_2|^2)\psi_1 - 2\mathrm{i}\psi_2^2\psi_3^* - \alpha(g^*_{1,1}g_{1,3} + g^*_{2,1}g_{2,3}),\\
\bar\partial\psi_2 &= -\mathrm{i}\partial^2\psi_2 - 2\mathrm{i}(|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2)\psi_2 - 2\mathrm{i}\psi_1\psi_2^*\psi_3 - \alpha(g^*_{1,1}g_{1,4} + g^*_{2,1}g_{2,4}),\\
\bar\partial\psi_3 &= -\mathrm{i}\partial^2\psi_3 - 2\mathrm{i}(2|\psi_2|^2 + |\psi_3|^2)\psi_3 - 2\mathrm{i}\psi_1^*\psi_2^2 - \alpha(g^*_{1,2}g_{1,4} + g^*_{2,2}g_{2,4}).
\end{aligned} \tag{17}$$

In deriving equation (17), we use the property $\sum_{l=1,4} g^*_{l,i} g_{l,j} = \delta_{i,j}$. The auxiliary equation (7) becomes

$$\begin{aligned} \partial g_{i,1} &= -g_{i,3}\psi_1^* - g_{i,4}\psi_2^*, & \partial g_{i,2} &= -g_{i,3}\psi_2^* - g_{i,4}\psi_3^*, \\ \partial g_{i,3} &= g_{i,1}\psi_1 + g_{i,2}\psi_2, & \partial g_{i,4} &= g_{i,1}\psi_2 + g_{i,2}\psi_3, \quad i = 1, 2. \end{aligned} \tag{18}$$

The generalized polarizations are defined by

$$g^{-1}\bar{T}g \equiv -\begin{pmatrix} \mathrm{i}D_1 & Q & P_1 & P_2 \\ -Q^* & \mathrm{i}D_2 & P_2 & P_3 \\ -P_1^* & -P_2^* & -\mathrm{i}D_1 & Q^* \\ -P_2^* & -P_3^* & -Q & -\mathrm{i}D_2 \end{pmatrix}, \tag{19}$$

where D_i, P_i are similarly defined as in equations (13), (14) such that $P_1 = i\alpha(g^*_{1,1}g_{1,3} + g^*_{2,1}g_{2,3})$, etc. The matrix form in equation (19) was introduced in [20], where it was argued that g should be skew-symmetric invariant as well as complex unitary. Equation (18) can be rewritten in terms of the generalized polarizations as,

$$\begin{aligned} \partial D_1 &= \mathrm{i}(P_1\psi_1^* - P_1^*\psi_1 + P_2\psi_2^* - P_2^*\psi_2), & \partial Q &= P_2^*\psi_1 - P_2\psi_3^* + P_3^*\psi_2 - P_1\psi_2^*, \\ \partial P_1 &= 2\mathrm{i}D_1^*\psi_1 + 2Q\psi_2, & \partial P_2 &= \mathrm{i}(D_1 + D_2)\psi_2 + Q\psi_3 - Q^*\psi_1. \end{aligned} \tag{20}$$

and similarly for D_2 with replacements $P_1 \to P_3$, $\psi_1 \to \psi_3$ in the first equation of (20), and for P_3 with replacements $D_1 \to D_2$, $\psi_1 \to \psi_3$, $Q \to -Q^*$ in the third equation of (20). It can be explicitly checked that $\partial\left(D_1^2 + D_2^2 + 2|Q|^2 + |P_1|^2 + 2|P_2|^2 + |P_3|^2\right) = 0$ by using equation (20), which results from $\mathrm{Tr}(g^{-1}\bar{T}g)^2 = -\alpha^2$.

3.3. Equation of $\frac{SO(8)}{U(4)}$ HSS

This model corresponds to taking $G = SO(8)$, $K = U(4)$, and E and T matrices are given by 8×8 matrices in equation (8) with $n = m = 4$, and

$$E_m = \begin{pmatrix} 0 & \psi_1 & \psi_3 & \psi_6 \\ -\psi_1 & 0 & \psi_2 & \psi_5 \\ -\psi_3 & -\psi_2 & 0 & \psi_4 \\ -\psi_6 & -\psi_5 & -\psi_4 & 0 \end{pmatrix}. \tag{21}$$

Then equation (5) becomes

$$\begin{aligned} \bar{\partial}\psi_1 &= -\mathrm{i}\partial^2\psi_1 - 2\mathrm{i}\sum_{i\neq 4}|\psi_i|^2\psi_1 - 2\mathrm{i}(-\psi_2\psi_6 + \psi_5\psi_3)\psi_4^* + \mathrm{i}P_1, \\ \bar{\partial}\psi_3 &= -\mathrm{i}\partial^2\psi_3 - 2\mathrm{i}\sum_{i\neq 5}|\psi_i|^2\psi_3 - 2\mathrm{i}(\psi_1\psi_4 + \psi_2\psi_6)\psi_5^* + \mathrm{i}P_3, \end{aligned} \tag{22}$$

where P_i are one of the generalized potentials defined by[3]

$$g^{-1}\bar{T}g \equiv -\begin{pmatrix} D & P \\ P^* & D^* \end{pmatrix}, \tag{24}$$

$$D = \begin{pmatrix} \mathrm{i}D_1 & Q_1 & Q_3 & Q_6 \\ -Q_1^* & \mathrm{i}D_2 & Q_2 & Q_5 \\ -Q_3^* & -Q_2^* & \mathrm{i}D_3 & Q_4 \\ -Q_6^* & -Q_5^* & -Q_4^* & \mathrm{i}D_4 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & P_1 & P_3 & P_6 \\ -P_1 & 0 & P_2 & P_5 \\ -P_3 & -P_2 & 0 & P_4 \\ -P_6 & -P_5 & -P_4 & 0 \end{pmatrix}. \tag{25}$$

[3] The matrix form in equation (24) was determined in [20] by requiring that $g \in U(8)$ as well as satisfying

$$g^T\begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}g = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}. \tag{23}$$

The explicit form of D_i, Q_i, P_i in terms of $g_{i,j}$ can be given similarly as in equations (13), (14). The auxiliary equation (6) in terms of the generalized potentials is

$$\begin{aligned}
\partial P_1 &= \mathrm{i}(D_1 + D_2)\psi_1 + Q_2\psi_3 + Q_5\psi_6 - Q_3\psi_2 - Q_6\psi_5,\\
\partial P_3 &= \mathrm{i}(D_1 + D_3)\psi_3 + Q_4\psi_6 - Q_2^*\psi_1 + Q_1\psi_2 - Q_6\psi_4,\\
\partial D_1 &= -2\,\mathrm{Im}(P_1\psi_1^* + P_3\psi_3^* + P_6\psi_6^*),\\
\partial Q_1 &= P_2^*\psi_3 + P_5^*\psi_6 - P_3\psi_2^* - P_6\psi_5^*,\\
\partial Q_3 &= -P_2^*\psi_1 + P_4^*\psi_6 + P_1\psi_2^* - P_6\psi_4^*.
\end{aligned} \tag{26}$$

Equations for other components can be obtained by applying the following substitutions on equation (26), $(\psi_1, P_1, Q_1, D_1) \to (\psi_2, P_2, Q_2, D_2) \to (\psi_4, P_4, Q_4, D_3) \to (-\psi_6, -P_6, -Q_6^*, D_4) \to (\psi_1, P_1, Q_1, D_1)$ and $(\psi_3, P_3, Q_3) \to (\psi_5, P_5, Q_5) \to (-\psi_3, -P_3, -Q_3^*) \to (-\psi_5, -P_5, -Q_5^*) \to (\psi_3, P_3, Q_3)$.

4. The Bäcklund transformation

4.1. Type-I BT

Let Ψ be the solution of the pair of the Lax equation (3) for a given $g \in G$. Here g is a solution of the SIT–NLS equation (5). We introduce a new solution $\hat{\Psi}$ in a form

$$\hat{\Psi} = (\lambda - \sigma)\Psi, \qquad \sigma = f^{-1}Mg, \tag{27}$$

where M is an arbitrary constant matrix. Here $f \in G$ is another new solution of the SIT–NLS equation (5). This relation (27) between $\Psi, \hat{\Psi}$ expressed by the two solutions f, g of the SIT–NLS equation is the Type-I BT [16]. This form is useful in deriving the permutability theorem, which then will be used in constructing two-soliton solutions [19].

4.2. Type-II BT

From the type-I BT, we can derive the BT (type-II BT) in a more familiar form. $\hat{\Psi}$ in equation (27) satisfies

$$\begin{aligned}
0 &= [\partial + F + \lambda T]\hat{\Psi} = [\partial + F + \lambda T](\lambda - \sigma)\Psi,\\
&= [-\partial\sigma - (\lambda - \sigma)(E + \lambda T) + (F + \lambda T)(\lambda - \sigma)]\Psi
\end{aligned} \tag{28}$$

where $F = f^{-1}\partial f$. $O(\lambda^0)$ of equation (28) gives $-\partial\sigma + \sigma E - F\sigma = 0$, which are satisfied as $\sigma = f^{-1}Mg$. $O(\lambda^1)$ of equation (28) gives

$$0 = F - E - [T, \sigma] = f^{-1}\partial f - g^{-1}\partial g - [T, f^{-1}Mg], \tag{29}$$

which becomes the first equation of the type-II BT.

Similarly, $\bar{\partial}$-part of the Lax equation (3) gives

$$\begin{aligned}
0 = \Bigg[&-\bar{\partial}\sigma + (\lambda - \sigma)\left(-\frac{1}{2}[E, \tilde{E}] + \partial\tilde{E} + \lambda E + \lambda^2 T + \frac{1}{\lambda}g^{-1}\bar{T}g\right)\\
&\times\left(\frac{1}{2}[F, \tilde{F}] - \partial\tilde{F} - \lambda F - \lambda^2 T - \frac{1}{\lambda}f^{-1}\bar{T}f\right)(\lambda - \sigma)\Bigg]\Psi.
\end{aligned} \tag{30}$$

$O(\lambda^{-1})$ of equation (30) gives $-\sigma g^{-1}\bar{T}g + f^{-1}\bar{T}f\sigma = 0$, which are satisfied by choosing $\sigma = f^{-1}Mg$ with $[\bar{T}, M] = 0$. $O(\lambda^0)$ of equation (30) gives

$$\left(f^{-1}\bar{\partial}f + \partial\tilde{F} - \tfrac{1}{2}[F, \tilde{F}]\right)\sigma - f^{-1}\bar{T}f = \sigma\left(g^{-1}\bar{\partial}g + \partial\tilde{E} - \tfrac{1}{2}[E, \tilde{E}]\right) - g^{-1}\bar{T}g, \tag{31}$$

which is the second equation of the type-II BT. $O(\lambda^1)$ of equation (30) gives

$$\tfrac{1}{2}[F, \tilde{F}] - \partial\tilde{F} + F\sigma = \tfrac{1}{2}[E, \tilde{E}] - \partial\tilde{E} + \sigma E, \tag{32}$$

which can be shown to be resulted from the first equation (29) of the type-II BT, see the appendix. $O(\lambda^2)$ of equation (30) gives once again the first equation (29) of the type-II BT.

Equations (29) and (31) give relations between f and g, and can be used to obtain a new solution f from a known solution g.

4.3. The equation of motion from the BT

The BT gives a method to obtain a new solution f from a known solution g. The consistency of the BT with the equation of motion means that once g is the solution of the equation of motion, then the new solution f should be the solution of the equation of motion, too. Here we give a proof of this statement. It shows typically how the property of the Hermitian symmetric space is involved in manipulating this type of equation.

We first introduce two identities, which are useful for our purpose. Let us start with

$$[T, \bar{\partial}\sigma] = [T, -f^{-1}\bar{\partial} f f^{-1} M g + f^{-1} M \bar{\partial} g]. \tag{33}$$

By applying the BT (31) on the right-hand side of equation (33), we obtain

$$\left[T, \partial\tilde{F}\sigma - \tfrac{1}{2}[F, \tilde{F}]\sigma - f^{-1}\bar{T} f - \sigma\partial\tilde{E} + \tfrac{1}{2}\sigma[E, \tilde{E}] + g^{-1}\bar{T} g\right]. \tag{34}$$

Using the properties of the HSS and applying the BT (29) on equation (34), we obtain

$$\begin{aligned}[T, \bar{\partial}\sigma] = {} & -\partial F\sigma + \partial\tilde{F}(F - E) - \tfrac{1}{2}[F, \tilde{F}](F - E) - [T, f^{-1}\bar{T} f] \\ & - (F - E)\partial\tilde{E} + \sigma\partial E + \tfrac{1}{2}(F - E)[E, \tilde{E}] + [T, g^{-1}\bar{T} g]. \end{aligned} \tag{35}$$

Finally applying equation (32) on equation (35) (and using the properties of HSS) gives

$$\begin{aligned}[T, \bar{\partial}\sigma] = {} & -\partial F\sigma + [\tilde{F}, \partial F] + \tfrac{1}{2}[F, [F, \tilde{F}]] - \tfrac{1}{2}[E, [E, \tilde{E}]] + \tfrac{1}{2}\partial[E, \tilde{E}] - 2F\sigma E \\ & + \sigma E^2 + F^2\sigma + \sigma\partial E + [T, g^{-1}\bar{T} g - f^{-1}\bar{T} f]. \end{aligned} \tag{36}$$

Equation (36) is the first identity.

The second identity is obtained by differentiating (∂) equation (32),

$$\tfrac{1}{2}\partial[F, \tilde{F}] - \partial^2\tilde{F} + \partial F\sigma - F^2\sigma + 2F\sigma E - \tfrac{1}{2}\partial[E, \tilde{E}] + \partial^2\tilde{E} - \sigma E^2 - \sigma\partial E = 0, \tag{37}$$

where we use the identity $\partial\sigma = -F\sigma + \sigma E$.

Finally, differentiating ($\bar{\partial}$) the BT (29) and using two identities in (36) and (37), we get

$$\bar{\partial} F - \bar{\partial} E - \tfrac{1}{2}[F, [F, \tilde{F}]] + \tfrac{1}{2}[E, [E, \tilde{E}]] + \partial^2\tilde{F} - \partial^2\tilde{E} - [T, g^{-1}\bar{T} g - f^{-1}\bar{T} f] = 0, \tag{38}$$

which proves our statement.

5. Discussion

In this paper, we have constructed multi-component SIT–NLS systems associated with the Hermitian symmetric spaces. These systems can describe matched pulses propagating through a Kerr medium doped with multi-level resonant atoms. The Hermitian symmetric spaces describe the SIT–NLS system naturally, where $g^{-1}\bar{T} g$ corresponds to the generalized population, while $g^{-1}\partial g$ describes the pulse propagation. The Lax pair is constructed in terms of these variables and the equation of motion is explicitly written for some specific HSSs. We introduce the BT of these systems, and show that they are compatible with the equation of motion. The complex structure of the HSS is found to be essential in manipulating all these relations.

The BT gives a natural way to find solitons of the multi-component SIT–NLS systems. Explicit construction of solitons using the BT will be given in a separate paper [19]. This type of construction is rare for multi-component integrable systems and should serve as a proper guidance to studies of more realistic systems.

Finally, we want to mention that our formalism can be easily extended to describe the SIT-higher derivative NLS system, where propagating pulses are very short or highly intensive [21]. An appropriate extension of the higher derivative NLS systems on Hermitian symmetric spaces in [11] should be conducted for this purpose.

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Appendix. Proof of equation (32)

We first prove the m-part of (32), which is

$$0 = \partial\tilde{F} - \partial\tilde{E} - (F\sigma - \sigma E)_m. \tag{A.1}$$

Here, $(F\sigma - \sigma E)_m$ and $(F\sigma - \sigma E)_k$ mean the m-part and the k-part of $(F\sigma - \sigma E)$, respectively. Using

$$-(F\sigma - \sigma E)_m = -(f^{-1}\partial f f^{-1} M g - f^{-1} M \partial g)_m = \partial(f^{-1} M g)_m = \partial\sigma_m, \tag{A.2}$$

the m-part of (32) can be written as

$$0 = \partial\tilde{F} - \partial\tilde{E} + \partial\sigma_m. \tag{A.3}$$

From the BT (29), we get

$$0 = \tilde{F} - \tilde{E} - [T, [T, \sigma]] = \tilde{F} - \tilde{E} + \sigma_m. \tag{A.4}$$

Now, we differentiate (∂) equation (A.4) to obtain equation (A.3).

The k-part of (32) is

$$0 = \tfrac{1}{2}[F, \tilde{F}] - \tfrac{1}{2}[E, \tilde{E}] + (F\sigma - \sigma E)_k. \tag{A.5}$$

Using the BT (29), equation (A.5) becomes

$$\begin{aligned} 0 &= \tfrac{1}{2}[E + [T, \sigma], \tilde{E} - \sigma_m] - \tfrac{1}{2}[E, \tilde{E}] + (E\sigma + [T, \sigma]\sigma - \sigma E)_k \\ &= -[E, \sigma_m] + [E, \sigma]_k - \tfrac{1}{2}[[T, \sigma], \sigma_m] + ([T, \sigma]\sigma)_k. \end{aligned} \tag{A.6}$$

Now, by noting

$$-\tfrac{1}{2}[[T, \sigma], \sigma_m] + ([T, \sigma]\sigma)_k = \tfrac{1}{2}[T, \sigma^2]_k = 0, \tag{A.7}$$

equation (A.5) is proved.

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